Additional Notes on First-Order Archimedean Generators

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1 Generator and Pseudo-inverse

1.1 Generator

$$\phi(u) = \frac{1-u}{u+\alpha} \tag{1}$$

with $u \in [0,1]$ and the parameter $\alpha \in [0,\infty)$.

1.2 Pseudo-inverse

$$\phi^{[-1]}(s) = \begin{cases} \dfrac{1-\alpha s}{1+s}, & s \leq \frac{1}{\alpha} \\ 0, & s \geq \frac{1}{\alpha} \end{cases} \tag{2}$$

2 Special Cases

- For $\alpha = 0$, the generator is strict, and can generate multivariate copulas
- As $\alpha \to \infty$, $\phi(u) \to 1-u$, which generates the lower bound coupla.

3 Level Sets

3.1 Zero set

The zero set of the copula generated by the strict generator is simply the u and v axes.

For non-strict generators, the zero set of the copula will be the region bounded by the u and v axes and the solution to:

$$\phi(0) = \phi(u) + \phi(v).$$

This may be solved for v as a function of u. Call this function $v_0(u)$:

$$v_0(u) = \frac{\alpha^2(1-u)}{\alpha^2 + u(1+2\alpha)} \tag{3}$$

After Nelsen, [p 123] this boundary of the zero set will be referred to as the *zero curve*.

3.2 Level sets for other copula values

The level sets for $0 < C(u, v) < 1$ are curves, solutions to:

$$\phi(v) = \phi(C) - \phi(v)$$

where C is the value of the copula for which the level set is desired. For some applications, it may be sufficient to express this in generic form:

$$v_C(u) = \phi^{[-1]}\left[\phi(C) - \phi(u)\right] \tag{4}$$

for $C \le u \le 1$ and $0 < C < 1$. However, it may be more convenient to have the level curves in closed form. First compute

$$s = \phi(C)$$

for the desired value of C. Obtain the result using:

$$v_C(u) = \frac{u\left[1 - \alpha(s+1)\right] + \alpha(2 - \alpha s)}{u(2+s) + \alpha(s+1) - 1} \tag{5}$$

for $u \le s$. (Like the generator itself, this is a first-order rational function in u.)

3.3 Level sets, graphically

Level curves for the strict generator ($\alpha = 0$) are presented in Figure 3.3 for $C = 1/8, 2/8, \ldots, 7/8$. The curve for each value of C passes through the points (1, C) and (C, 1).

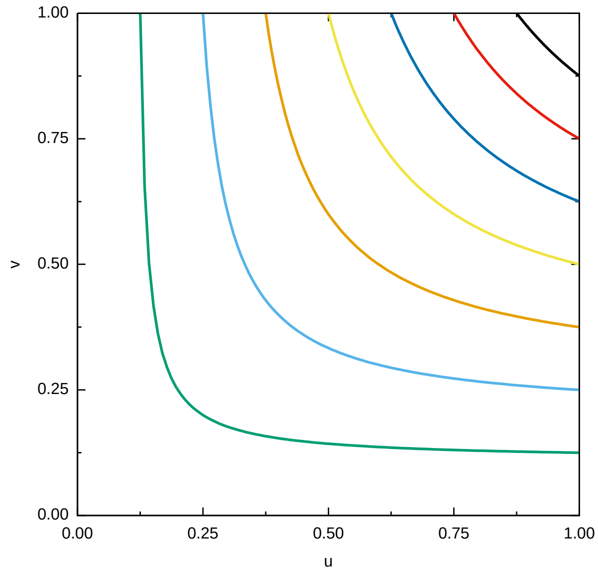

Figure 1: Level sets, $\alpha = 0$

Level sets for $\alpha = 0$, in increments of 1/8, starting at 1/8 and going up to 7/8. Because the generator for $\alpha = 0$ is strict, the zero set is simply the u and v axes.

Level curves for selected non-zero values of α are presented in Figure 3.3 for the same values of C as in Figure 3.3. In addition, the curve in each sub-figure passing through the points (1, 0) and (0, 1) are the zero curves for that value of α.

In the last subfigure in Figure 3.3 (α =10), the level curves are nearly lines, indicating this copula is very close to the lower-bound copula W.

4 Derivatives; copula density

Derivatives of the copula have several applications. These include random pair generation, iterative optimization of the parameter to a data set, and the copula density. In this section, the partial derivatives are presented, together with the mixed partial derivative.

4.1 Derivative of generator

$$\frac{d\phi(u)}{du} = -\frac{1+\alpha}{(u+\alpha)^2} \tag{6}$$

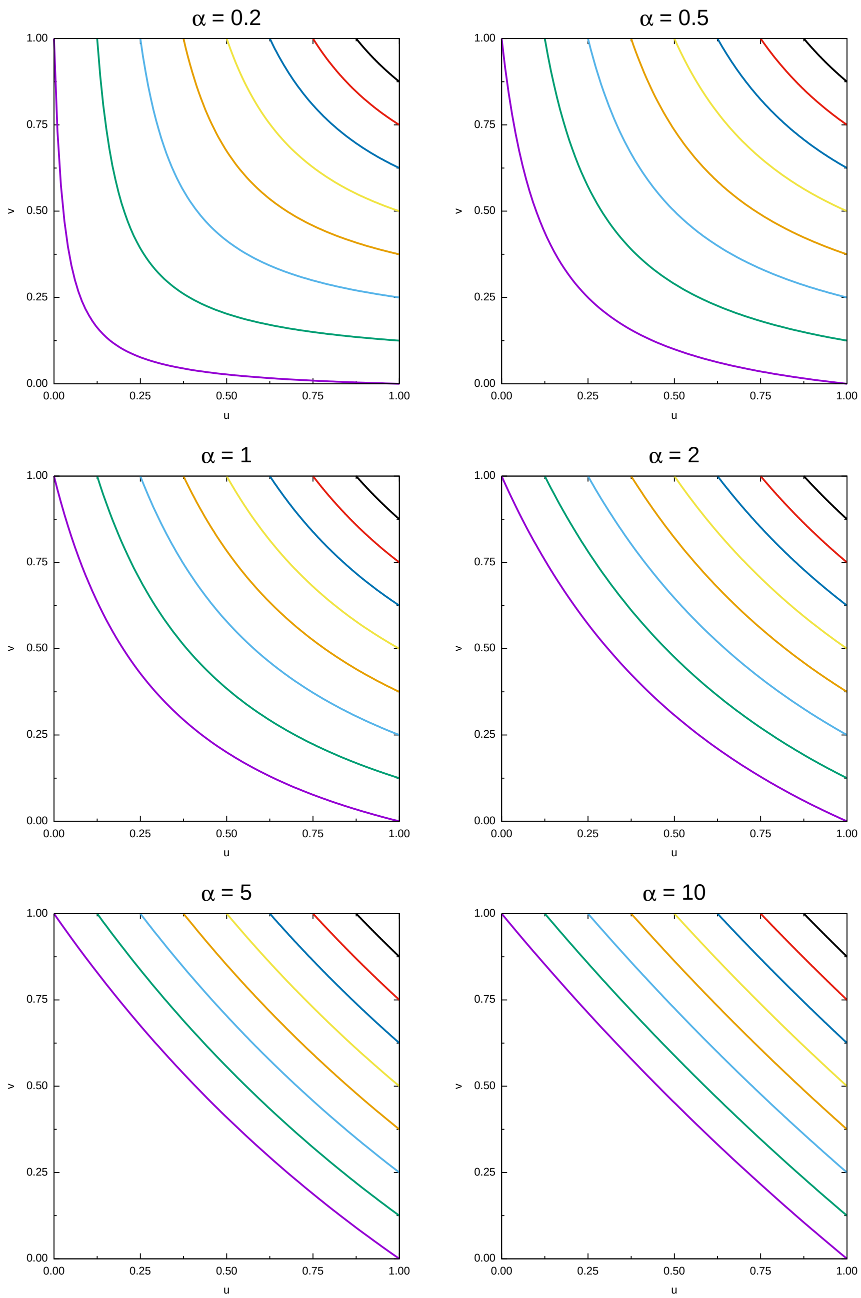

Figure 2: Level sets, $\alpha \in \{0.2, 0.5, 1, 2, 5, 10\}$

The zero set in each plot is bounded by the u *and* v *axes and the leftmost curve. The remaining curves in each plot are the level sets for* $C(u, v) = 1/8, 2/8, \ldots, 7/8$.

4.2 Derivative of inverse generator

The derivative of the inverse generator exists for positive reals, other than $\frac{1}{\alpha}$:

$$\frac{d\phi^{[-1]}}{ds} = \begin{cases} -\dfrac{1+\alpha}{(1+s)^2}, & 0 < s < \frac{1}{\alpha} \\ 0, & s > \frac{1}{\alpha} \end{cases} \tag{7}$$

4.3 Partial derivative of copula

Recognizing that $s = \phi(u)+\phi(v)$, the equations in the previous two sections may be combined:

$$\frac{\partial C(u,v)}{\partial u} = \begin{cases} \left[\dfrac{1+\alpha}{(u+\alpha)(1+s)}\right]^2, & s < \frac{1}{\alpha} \\ 0, & s > \frac{1}{\alpha} \end{cases} \tag{8}$$

Because the variables u and v are exchangeable in an Archimedean copula, the partial derivative with respect to v is analogous.

4.4 Second mixed partial derivative

$$\frac{\partial^2 C}{\partial u \partial v} = \begin{cases} 2\left(\dfrac{1+\alpha}{1+s}\right)^3 (u+\alpha)^{-2}(v+\alpha)^{-2}, & s < 1/\alpha \\ 0, & s > 1/\alpha \end{cases}$$

Making the substitution

$$1+s = \frac{u+v-uv+\alpha(2+\alpha)}{(u+\alpha)(v+\alpha)}$$

in the previous equation yields:

$$\frac{\partial^2 C}{\partial u \partial v} = \begin{cases} 2\left(\dfrac{1+\alpha}{u+v-uv+\alpha(2+\alpha)}\right)^3 (u+\alpha)(v+\alpha), & s < 1/\alpha \\ 0, & s > 1/\alpha \end{cases} \tag{9}$$